ISRAEL JOURNAL OF MATHEMATICS 80 (1992), 161–169

BEREZIN INTEGRATION ON DEWITT SUPERMANIFOLDS*

BY

U. BRUZZO

Department of Mathematics, University of Genova
Via L. B. Alberti 4, 16132 Genova, Italy
e-mail: bruzzo@matgen.ge.cnr.it

ABSTRACT

We show that Berezin integration can be consistently defined on a supermanifold $(M, \mathcal{A})$ of the DeWitt type. The resulting integral is a real-valued functional on the space of compactly supported global sections of the Berezinian sheaf of $(M, \mathcal{A})$.

1. Introduction and preliminaries

Berezin integration on a graded manifold $(X, \mathcal{A})$ is a functional over the real vector space of compactly supported global sections of the Berezinian sheaf of $(X, \mathcal{A})$ (by "graded manifolds" we mean the Berezin-Leĭtes-Kostant supermanifolds [3,6]). In [5] an intrinsic definition of the Berezin integral was given; in that picture, the Berezinian sheaf is realized as $\Omega^m_{\mathcal{A}} \otimes \mathcal{P}_n/\mathcal{K}$, where $\Omega^m_{\mathcal{A}}$ is the sheaf of graded differential m-forms, $\mathcal{P}_n$ is the sheaf of graded differential operators of order n on $\mathcal{A}$, $\mathcal{K}$ is a suitable subsheaf of $\Omega^m_{\mathcal{A}} \otimes \mathcal{P}_n$, and $(m, n) = \dim(X, \mathcal{A})$.

Let $(M, \mathcal{A})$ be a supermanifold of the DeWitt type, based on a graded-commutative Banach algebra B (the latter can also be infinite-dimensional, although in that case it must satisfy additional requirements). We work with a category of supermanifolds that has been described in [2] and is strictly related to Rothstein's supermanifolds [10]; it can be regarded as a natural evolution of the notion of

* Research partly supported by 'Gruppo Nazionale per la Fisica Matematica' of CNR and by the Italian Ministry for the Universities and Scientific and Technological Research through the research project 'Metodi geometrici in relatività e teorie di campo.'
Received April 15, 1992

supermanifold as introduced by DeWitt [4] and Rogers [9]. This category has been studied in some detail in [1] in the case of a finite-dimensional B.

Let us recall that a supermanifold of the DeWitt type $(M, \mathcal{A})$ (we shall define precisely this notion in the next Section), say of dimension (m, n), determines an m-dimensional smooth ordinary manifold $\widetilde{M}$, called the **body manifold** of $(M, \mathcal{A})$, with a smooth projection $\Phi: M \to \widetilde{M}$. We shall show that a Berezin integral can be defined on $(M, \mathcal{A})$ in a natural way; it integrates compactly supported global sections of a suitably defined Berezinian sheaf to yield a real number. That this can be done is no surprise, in that there is a one-to-one correspondence between isomorphism classes of DeWitt supermanifolds with a fixed body manifold $\widetilde{M}$ and isomorphism classes of graded manifolds based on $\widetilde{M}$ [1]. However, this is not an equivalence of categories: DeWitt supermanifolds have more morphisms than graded manifolds. It turns out that the Berezin integral over a DeWitt supermanifold $(M, \mathcal{A})$ is invariant under the full group of diffeomorphisms of $(M, \mathcal{A})$.

It should be stressed that the integral so defined, when evaluated on a section of the Berezinian sheaf, yields a real number. To our knowledge it is not possible — contrary to some claims raised in the literature — to define consistently a B-valued integral on DeWitt supermanifolds.

Let us now give some basic definitions; a survey of the algebraic foundations of supermanifold theory can be found in [1]. We shall consider $\mathbb{Z}_2$-graded objects; for simplicity, we shall say "graded" instead of "$\mathbb{Z}_2$-graded." A graded algebra $B = B_0 \oplus B_1$ is said to be graded-commutative if

$$ab = (-1)^{\alpha\beta} ba \qquad \text{whenever} \qquad a \in B_\alpha,\ b \in B_\beta, \qquad \alpha, \beta \in \mathbb{Z}_2\,.$$

The degree of a homogeneous element $a \in B$ will denoted by $|a|$. We shall allow B to be infinite-dimensional, but requiring that B is local and that the linear span of products of odd elements is dense in the radical $\mathfrak{Rad}\ B$ of B. We shall call the algebras satisfying these requirements **BGO-algebras**, meaning "Banach algebras of Grassmann origin." Alternative definitions of these algebras, and examples, can be found in [2,7,8].

Let us denote by σ the projection $B \to B/\mathfrak{Rad} B \simeq \mathbb{R}$ (the "body map"). The (m, n) dimensional "superspace" $B^{m,n}$ is the B_0-module $B_0^m \times B_1^n$; it can be endowed either with the vector space topology (sometimes called the "fine topology") or the **DeWitt (coarse) topology**, i.e. the topology whose open sets are the counterimages of the open sets in $\mathbb{R}^m$ via the projection $\sigma^{m,n}\colon B^{m,n} \to \mathbb{R}^m$ given by

$$(x^1, \ldots, x^m, y^1, \ldots, y^n) \mapsto (\sigma(x^1), \ldots, \sigma(x^m)).$$

A **graded ringed B-space** is a pair $(X, \mathcal{A})$, with X a topological space and $\mathcal{A}$ a sheaf of graded-commutative B-algebras on X. A graded ringed space is said to be *local* if each stalk $\mathcal{A}_x$ is a local graded ring, i.e. if it contains a unique maximal graded ideal. The **sheaf $\mathcal{D}er\mathcal{A}$ of derivations** of $\mathcal{A}$ is the sheaf associated with the presheaf of $\mathcal{A}$-modules $U \mapsto \{\text{graded derivations of } \mathcal{A}_{|U}\}$; a graded derivation of $\mathcal{A}_{|U}$ is an endomorphism of sheaves of graded B-algebras $D\colon \mathcal{A}_{|U} \to \mathcal{A}_{|U}$ satisfying the graded Leibniz rule $D(a \cdot b) = D(a) \cdot b + (-1)^{|a||D|} a \cdot D(b)$.

Finally, $\mathcal{D}er^*\mathcal{A}$ denotes the dual sheaf to $\mathcal{D}er\,\mathcal{A}$, i.e.

$$\mathcal{D}er^*\mathcal{A} = \mathcal{H}om_{\mathcal{A}}(\mathcal{D}er\,\mathcal{A}, \mathcal{A}).$$

The **exterior differential** is the morphism of sheaves of graded B-modules $d\colon \mathcal{A} \to \mathcal{D}er^*\mathcal{A}$ defined by $df(D) = (-1)^{|f||D|} D(f)$ for all homogeneous $f \in \mathcal{A}(U)$, $D \in \mathcal{D}er\,\mathcal{A}(U)$ and all open $U \subset M$.

2. Supermanifolds of the DeWitt type

We introduce some elements that we shall use to define DeWitt supermanifolds. Let B be a BGO-algebra. For each pair of nonnegative integers (m, n) we consider the B_0-module $B^{m,n}$ endowed with the DeWitt topology.* Let $U \subset B^{m,0}$ be an open set; a smooth map $f\colon U \to B$ is said to be G^∞ [9,2] if its Fréchet differential is B_0-linear. A G^∞ function $f(x,y)$ on $B^{m,n}$ is a smooth map that can be written in the form

$$f(x,y) = f_0(x) + \sum_{\alpha=1}^{n} f_\alpha(x)\, y^\alpha + \cdots + f_{1\ldots n}\, y^1 \cdots y^n \tag{2.1}$$

for some (not uniquely defined) G^∞ functions $f_{\ldots}(x)$. The sheaf of G^∞ functions on $B^{m,0}$ (resp. $B^{m,n}$) will be denoted by $\hat{\mathcal{G}}^\infty$ (resp. $\mathcal{G}^\infty$).

Let Λ_n be the exterior algebra generated over $\mathbb{R}$ by the elements $y^1 \ldots y^n$ in B_1. We define a sheaf $\mathcal{G}$ on $B^{m,n}$ by letting $\mathcal{G} = p^{-1}\hat{\mathcal{G}}^\infty \otimes_{\mathbb{R}} \Lambda_n$, where $p\colon B^{m,n} \to B^{m,0}$ is the natural projection. Moreover, let $ev\colon \mathcal{G} \to \mathcal{C}$, where $\mathcal{C}$

* If B is finite-dimensional, say is an exterior algebra with L generators, then for consistency one must assume $L \geq n$.

is the sheaf of continuous B-valued functions on $B^{m,n}$, be the graded B-algebra morphism defined by $ev(f \otimes \xi) = f\xi$; one has $\operatorname{Im} ev \simeq \mathcal{G}^\infty$. The pair $(B^{m,n}, \mathcal{G})$ is a local graded B-space, and the maximal graded ideal of $\mathcal{G}_z$ is

$$\mathfrak{L}_z = \{f \in \mathcal{G}_z \text{such that} ev(f)(z) = 0\}.$$

DEFINITION 2.1: *The standard supermanifold over $B^{m,n}$ is the triple $(B^{m,n}, \mathcal{G}, ev)$.*

The graded algebras $\mathcal{G}(U)$, where U is an open set in $B^{m,n}$, can be topologized by means of the family of seminorms $p_{L,K}: \mathcal{G}(U) \to \mathbb{R}$ defined by

$$p_{L,K}(f) = \max_{z \in K} \| \, ev\big(L(f)(z)\big) \, \| , \tag{2.2}$$

where L runs over the differential operators of $\mathcal{G}$ on U, K runs over the compact subsets of U, and $\| \; \|$ is the norm in B (cf. [6,2]). In this way $\mathcal{G}(U)$ becomes a locally convex topological $\mathbb{R}$-algebra.

PROPOSITION 2.2 [2]: *The topological algebra $\mathcal{G}(U)$ is complete for every open set $U \subset B^{m,n}$.*

Thus, each $\mathcal{G}(U)$ is a Fréchet graded B-algebra.

We can eventually supply the definition of a DeWitt supermanifold. Let M be a second countable topological space, $\mathcal{A}$ a sheaf of graded-commutative B-algebras on M, and let $ev^M : \mathcal{A} \to \mathcal{C}_M$ be a morphism of sheaves of graded B-algebras ($\mathcal{C}_M$ is the sheaf of continuous B-valued functions on M). We say that the triple $(M, \mathcal{A}, ev^M)$ is an (m,n) dimensional DeWitt supermanifold if it is locally isomorphic to the standard supermanifold over $B^{m,n}$; more precisely, we have the following definition.

DEFINITION 2.3: *The triple $(M, \mathcal{A}, ev^M)$ is an (m,n) dimensional DeWitt supermanifold if each point $z \in M$ has a neighborhood U endowed with a pair $(f, f^\sharp)$, where*

(1) *$f: U \to B^{m,n}$ is a homeomorphism onto an open subset of $B^{m,n}$;*
(2) *$f^\sharp: \mathcal{G}_{|f(U)} \to f_*(\mathcal{A}_{|U})$ is an isomorphism of sheaves of graded B-algebras, such that the diagram*

$$\begin{array}{ccc} \mathcal{G}_{|f(U)} & \xrightarrow{f^\sharp} & f_*(\mathcal{A}_{|U}) \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev^M} \\ \mathcal{C}_{|f(U)} & \xrightarrow[f^*]{} & f_*(\mathcal{C}_{M|U}) \end{array}$$

commutes.

DEFINITION 2.4: *A morphism of DeWitt supermanifolds $(M, \mathcal{A}, ev^M) \to (N, \mathcal{B}, ev^N)$ is a pair $(f, f^\sharp)$, where $f\colon M \to N$ is a continuous mapping, and $f^\sharp\colon \mathcal{B} \to f_*\mathcal{A}$ is a morphism of sheaves of graded B-algebras such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f^\sharp} & f_*\mathcal{A} \\ {\scriptstyle ev^N}\downarrow & & \downarrow{\scriptstyle ev^M} \\ \mathcal{C}_N & \xrightarrow[f^*]{} & f_*\mathcal{C}_M \end{array}$$

One can prove [2] that f is a G^∞ morphism, and that for each open set $U \subset N$, the morphism $f_U^\sharp\colon \mathcal{B}(U) \to \mathcal{A}(f^{-1}(U))$ is continuous. We say that $(f, f^\sharp)$ is a diffeomorphism if $f^\sharp$ is a sheaf isomorphism; this implies that f is invertible and that f^{-1} is G^∞.

Remarks:

(1) In view of the local isomorphism with the standard supermanifold, the algebras $\mathcal{A}(U)$ are Fréchet graded B-algebras.

(2) Since $B^{m,n}$ has been topologized with the DeWitt (coarse) topology, the Definition 2.2 does not yield a generic supermanifold, but rather a DeWitt supermanifold. In other terms, the choice of the DeWitt topology is the origin of the structural results that we are going to discuss. ∎

Let $\mathcal{A}^\infty = ev^M(\mathcal{A})$. Then $\mathcal{A}^\infty$ endows M with a structure of G^∞ supermanifold [10,2]. Let $\mathfrak{A} = \{(U_i, \psi_i)\}_{i\in\mathbb{N}}$ be a G^∞ atlas for M. We establish in M the following equivalence relation [9]: two points $p, q \in M$ are in relation if there is an $i \in \mathbb{N}$ such that $p, q \in U_i$, and moreover $\sigma^{m,n}(\psi_i(p)) = \sigma^{m,n}(\psi_i(q))$; the quotient under this relation is an m-dimensional smooth manifold $\widetilde{M}$, and the projection $\Phi\colon M \to \widetilde{M}$ is smooth as well. $\widetilde{M}$ is often called the **body manifold** of M.

Another important result is the following: there is a graded manifold $(\widetilde{M}, \mathcal{F})$, based on $\widetilde{M}$, such that $\mathcal{A} \simeq \Phi^{-1}\mathcal{F} \otimes_{\mathbb{R}} B$. On this basis one can prove that there is a one-to-one correspondence between isomorphisms classes of (m,n) dimensional DeWitt supermanifolds with a fixed body manifold $\widetilde{M}$, and isomorphism classes of (m,n) dimensional graded manifolds based on $\widetilde{M}$. However, it turns out that the category of DeWitt supermanifold has more morphisms than the category of graded manifolds; the following Example shows an endomorphism of a DeWitt supermanifold that does not come from an endomorphism of the corresponding graded manifold.

Example: Let $(B^{1,1}, \mathcal{G}, ev)$ be the standard supermanifold over $B^{1,1} \equiv B$, and let f: $B^{1,1} \to B^{1,1}$ be the map $f(x,y) = (ax, y)$ with $a \in (\mathfrak{Nad}B)_0$. Since $\mathcal{G} = p^{-1}\hat{\mathcal{G}}^\infty \otimes_{\mathbb{R}} \Lambda_1$ then a section g of $\mathcal{G}$ has the form $g = g_0 + g_1 \otimes y$ where the g_i's are G^∞ functions of x. We define a sheaf morphism $f^\sharp$: $\mathcal{G} \to f_*\mathcal{G}$ by letting

$$f^\sharp(g) = f^*(g_0) + f^*(g_1) \otimes y\,.$$

The pair $(f, f^\sharp)$ is a morphism $(B^{1,1}, \mathcal{G}, ev) \to (B^{1,1}, \mathcal{G}, ev)$ that does not correspond to any morphism of the underlying graded manifold $(\mathbb{R}, C^\infty_{\mathbb{R}} \otimes \Lambda_1)$. ∎

Once more, we would like to stress that these structural results hold because DeWitt supermanifolds are modelled on the standard supermanifold over $B^{m,n}$, this space being endowed with the DeWitt topology. If $B^{m,n}$ is equipped with the fine topology, one obtains a wider category of supermanifolds, that may not admit a body manifold, and are not directly related to graded manifolds.

3. Berezin integration

Let $(M, \mathcal{A}, ev^M)$ be a DeWitt supermanifold. The sheaf $\mathcal{D}er\mathcal{A}$ of derivations of $\mathcal{A}$, and the dual sheaf $\mathcal{D}er^*\mathcal{A}$, are locally free. This implies that the sheaf $\Omega^k_{\mathcal{A}} = \wedge^k_{\mathcal{A}}\mathcal{D}er^*\mathcal{A}$ (the sheaf of differential k-superforms) is locally free as well. Here $\wedge_{\mathcal{A}}$ denotes the graded wedge product over $\mathcal{A}$.

We may define a morphism $\sim\, : \Omega^k_{\mathcal{A}} \to \Omega^k_{\widetilde{M}}$ (where $\Omega^k_{\widetilde{M}}$ is the sheaf of smooth differential forms on $\widetilde{M}$) as follows. Let f be a section of $\mathcal{A}$. Then the real-valued function $\tilde{f}(p) = \sigma(ev^M(f)(p))$ is constant along the fibers of Φ: $M \to \widetilde{M}$, and determines a function on $\widetilde{M}$, that we denote by the same symbol. This induces the required morphism $\sim\, : \Omega^k_{\mathcal{A}} \to \Omega^k_{\widetilde{M}}$. One should notice that in general it is not possible to define a morphism that maps the differential superforms on $(M, \mathcal{A}, ev^M)$ to B-valued differential forms on $\widetilde{M}$, and this is the ultimate reason why Berezin integration is real-valued and not B-valued.

Let $\mathcal{P}_k$ be the sheaf of graded differential operators of order k on $\mathcal{A}$. In addition to the left $\mathcal{A}$-module structure given by the product $(f \cdot D)(g) = fD(g)$, the sheaf $\mathcal{P}_k$ has a right $\mathcal{A}$-module structure, inequivalent to the previous one, according to the rule $(D \cdot f)(g) = D(fg)$. We consider in $\mathcal{P}_k$ this second $\mathcal{A}$-module structure, and take the graded tensor product $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{P}_n$, where $(m,n) = \dim(M, \mathcal{A}, ev^M)$.

The module $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{P}_n$ has gotten a distinguished graded submodule $\mathcal{K}$, whose sections ω are such that for any compactly supported section f of $\mathcal{A}$ there is a compactly supported section η of $\Omega^{m-1}_{\widetilde{M}}$ such that

$$\widetilde{\omega(f)} = d\eta\,.$$

The quotient $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{P}_n/\mathcal{K}$ is by definition the *Berezinian sheaf* $\mathcal{B}er(M,\mathcal{A})$ of $(M, \mathcal{A}, ev^M)$. It is a locally free graded $\mathcal{A}$-module of rank (1,0) (resp. (0,1)) if n is even (resp. odd), namely, $\mathcal{B}er(M,\mathcal{A})$ is the sheaf of sections of a superline bundle [1]. If $(x^1, \dots, x^m, y^1, \dots, y^n)$ is a set of local coordinates for $(M, \mathcal{A}, ev^M)$, the section

$$\left[(dx^1 \wedge \cdots \wedge dx^m) \otimes \frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^n}\right]$$

is a local basis for $\mathcal{B}er(M,\mathcal{A})$; here the square brackets denote the equivalence class with respect to the quotient by $\mathcal{K}$. All this is proved as in the case of graded manifolds [5].

We may now define the Berezin integral; we assume the body manifold $\widetilde{M}$ of $(M, \mathcal{A}, ev^M)$ is compact and oriented. Let ω be a global section of $\mathcal{B}er(M,\mathcal{A})$, and let ξ be a section of $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{P}_n$ such that $[\xi] = \omega$. Then we set

$$\int_{(M,\mathcal{A})} \omega = \int_M \widetilde{\xi(1)}\,;$$

here $\int_M$ is the usual integral of m-forms on an oriented m-dimensional manifold. The coordinate representation of this Berezin integral is the usual one: if ω is a section of $\mathcal{B}er(M,\mathcal{A})$ whose compact support is in the domain of a coordinate system $(x^1, \dots, x^m, y^1, \dots, y^n)$, and

$$\omega = \left[(dx^1 \wedge \cdots \wedge dx^m) \otimes \frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^n}\right] f\,,$$

then

$$\int_{(M,\mathcal{A})} \omega = \int f_{1\dots n}\, dx^1 \dots dx^m\,,$$

where $f_{1\dots n}$ is the last term in the "superfield expansion" (2.1).

Finally, we wish to check the invariance of the Berezin integral under diffeomorphisms of $(M, \mathcal{A}, ev^M)$. To this end we need the following result.

LEMMA 3.1: *Let $(f, f^\sharp)$ be a morphism $(M, \mathcal{A}, ev^M) \to (M, \mathcal{A}, ev^M)$. Then there is a unique smooth morphism $\tilde{f}\colon \widetilde{M} \to \widetilde{M}$ such that the following two diagrams commute:*

$$
(3.1)\qquad
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & M \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\widetilde{M} & \xrightarrow[\ \tilde{f}\ ]{} & \widetilde{M}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega^k_{\mathcal{A}} & \xrightarrow{\ f^\sharp\ } & f_*\Omega^k_{\mathcal{A}} \\
{\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\
\Omega^k_{\widetilde{M}} & \xrightarrow[\ \tilde{f}^*\ ]{} & f_*\Omega^k_{\widetilde{M}}
\end{array}
\qquad k \geq 0.
$$

Proof: Let us first consider the case of the standard supermanifold $(B^{m,n}, \mathcal{G}, ev)$ over $B^{m,n}$. Here the existence of $\tilde{f}$ follows from the following facts:

(i) $\mathcal{G}$ is isomorphic with the sheaf of supermanifold morphisms from $(B^{m,n}, \mathcal{G}, ev)$ into the standard supermanifold over $B^{1,1}$;

(ii) any section of $\mathcal{G}$ can be uniformly approximated by polynomials in the coordinates $(x^1, \ldots, x^m, y^1, \ldots, y^n)$.

The result is then extended to a generic DeWitt supermanifold $(M, \mathcal{A}, ev^M)$ because one can choose a G^∞ atlas $\mathfrak{A} = \{(U_i, \psi_i)\}$ on M, and a C^∞ atlas $\widetilde{\mathfrak{A}} = \{(V_i, \phi_i)\}$ on $\widetilde{M}$ such that $V_i = \Phi(U_i)$, and the diagram

$$
\begin{array}{ccc}
U_i & \xrightarrow{\ \psi_i\ } & B^{m,n} \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \sigma^{m,n}} \\
V_i & \xrightarrow[\ \phi_i\ ]{} & \mathbb{R}^m
\end{array}
$$

commutes. The commutativity of the two diagrams (3.1) can be checked directly. The uniqueness of $\tilde{f}$ follows from the commutativity of the first diagram. ∎

COROLLARY 3.2: *The sheaf $\mathcal{K}$ is invariant under any morphism $(f, f^\sharp)\colon (M, \mathcal{A}, ev^M) \to (M, \mathcal{A}, ev^M)$, in the sense that $f^\sharp(\mathcal{K})$ is a subsheaf of $f_*\mathcal{K}$.*

Let $(f, f^\sharp)$ be a diffeomorphism $(M, \mathcal{A}, ev^M) \to (M, \mathcal{A}, ev^M)$. The Berezinian superdeterminant [1,3,4] of the morphism $f^\sharp\colon f_*\Omega^1_{\mathcal{A}} \to \Omega^1_{\mathcal{A}}$ yields a globally defined section $\mathrm{Ber}(f^\sharp)$ of $\mathcal{A}$.

If $\widetilde{M}$ is oriented, we say that $(f, f^\sharp)$ is **orientation-preserving** if $\widetilde{\mathrm{Ber}(f^\sharp)}$ is a positive function on $\widetilde{M}$. Notice that this does not imply that $\tilde{f}$ is orientation-preserving: $\tilde{f}$ might reverse the orientation of $\widetilde{M}$, but this effect might be compensated by a change of sign of the degree n component in the development (2.1).

A diffeomorphism $(f, f^\sharp)$ of $(M, \mathcal{A}, ev^M)$ acts in a natural way on $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{P}_n$, and, in view of Corollary 3.2, this action descends to $\mathcal{B}er(M, \mathcal{A})$. Moreover, Corollary 3.2, and the invariance of the ordinary integral on smooth manifolds under orientation-preserving diffeomorphisms, allows one to prove the invariance of the Berezin integral:

THEOREM 3.3: *Let $(f, f^\sharp)$ be an orientation-preserving diffeomorphism of $(M, \mathcal{A}, ev^M)$, and let ω be a section of $\mathcal{B}er(M, \mathcal{A})$ with compact support. Then one has*

$$\int_{(M,\mathcal{A})} \omega = \int_{(M,\mathcal{A})} f^\sharp(\omega) .$$

ACKNOWLEDGEMENT: I wish to thank V.G. Pestov for valuable suggestions.

References

[1] C. Bartocci, U. Bruzzo and D. Hernández Ruipérez, *The geometry of supermanifolds,* Kluwer Acad. Publ., Dordrecht, 1991.

[2] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez and V. G. Pestov, *Foundations of supermanifold theory: the axiomatic approach,* Preprint 194/1992, Dipartimento di Matematica, Università di Genova; to appear in Diff. Geom. Appl.

[3] F. A. Berezin and D. A. Leĭtes, *Supermanifolds,* Soviet Math. Dokl. **16** (1975), 1218–1222.

[4] B. S. De Witt, *Supermanifolds,* Cambridge University Press, London, 1984.

[5] D. Hernández Ruipérez and J. Muñoz Masqué, *Construction intrinsèque du faisceau de Berezin d'une variété graduée,* C. R. Acad. Sci. Paris **301** (1985), 915–918.

[6] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization,* in *Differential Geometric Methods in Mathematical Physics,* Lecture Notes Math. **570** (K. Bleuler and A. Reetz, eds.), Springer-Verlag, Berlin, 1977, pp. 177–306.

[7] V. G. Pestov, *Ground algebras for superanalysis,* Rep. Math. Phys. **29 (3)** (1991), to appear.

[8] V. G. Pestov, *General construction of Banach–Grassmann algebras,* Seminario **179**, Dipartimento di Matematica, Università di Genova, 1991.

[9] A. Rogers, *A global theory of supermanifolds,* J. Math. Phys. **21** (1980), 1352–1365.

[10] M. Rothstein, *The axioms of supermanifolds and a new structure arising from them,* Trans. Amer. Math. Soc. **297** (1986), 159–180.